CFT & gravity week 6

OPE - scalar case

a) $$\mathcal{O}'(x) = \lambda^{\Delta}\,\mathcal{O}(x') \quad (x' = \lambda x)$$ (valid also for spinning operators)

Plug this in the OPE:

$$(1)\quad \mathcal{O}_1'(x)\,\mathcal{O}_2'(0) \sim \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta+\ell}}\, F^{(12k)}_{a_1\ldots a_\ell}(x,\partial_y)\, \mathcal{O}'^{a_1\ldots a_\ell}_k(y)\Big|_{y=0}$$

$$\downarrow$$

$$\lambda^{\Delta_1+\Delta_2}\,\mathcal{O}_1(\lambda x)\,\mathcal{O}_2(0) \sim \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta+\ell}}\, F^{\cdots}_{\cdots}(x,\partial_y)\,\lambda^{\Delta}\,\mathcal{O}_k^{a_1\ldots a_\ell}(\lambda y)\Big|_{y=0}$$

Rename $x \to \frac{x}{\lambda}$ $\quad y \to \frac{y}{\lambda}$

$$(3)\quad \mathcal{O}_1(x)\,\mathcal{O}_2(0) \sim \lambda^{-\Delta_1-\Delta_2+\Delta}\,\lambda^{\Delta_1+\Delta_2-\Delta+\ell}\,\frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta+\ell}}$$

$$F^{(12k)}_{a_1\ldots a_\ell}(\lambda^{-1}x,\lambda\partial_y)\,\mathcal{O}_k^{a_1\ldots a_\ell}(y)\Big|_{y=0}$$

Invariance implies that (1) and (3) have the same form, so

$$F^{(12k)}_{a_1\ldots a_\ell}(\lambda^{-1}x,\lambda\partial_y) = \lambda^{-\ell}\,F^{(12k)}_{a_1\ldots a_\ell}(x,\partial_y)\,.$$

b) Plugging the $\mathcal{O}_1 \mathcal{O}_2$ OPE in the three-point function

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle$$

only the conformal family of $\mathcal{O}_3$ survives.
We immediately find that the OPE coefficient c_{123} is also the coefficient of the three point function. Furthermore:

$$\cancel{\frac{1}{|x|^{\Delta_1+\Delta_2-\Delta_3}}} F^{(123)}(x, \partial_y) \langle \mathcal{O}_3(y) \mathcal{O}_3(w) \rangle \Big|_{y=0} =$$

$$= \frac{1}{\cancel{|x|^{\Delta_1+\Delta_2-\Delta_3}} |w|^{\Delta_3+\Delta_2-\Delta_1} |x-w|^{\Delta_1+\Delta_3-\Delta_2}}$$

So:
$$F^{(123)}(x, \partial_y) \frac{1}{(y^2 - 2y\cdot w + w^2)^{\Delta_3}} \Big|_{y=0} = \frac{1}{|w|^{2\Delta_3}} \left(\frac{|x-w|}{|w|} \right)^{\Delta_2 - \Delta_1 - \Delta_3}$$

$$F^{(123)}(x, \partial_y) \left(1 + \frac{y^2 - 2y\cdot w}{w^2} \right)^{-\Delta_3} \Big|_{y=0} = \left(1 + \frac{x^2 - 2x\cdot w}{w^2} \right)^{\frac{\Delta_2-\Delta_1-\Delta_3}{2}}$$

c) and d) see Mathematica code.

Radial quantization of a free scalar

1. The tools you learn in GR are a great bookkeeping device for changing coordinates:

$$S_E = \int d^dx \, \frac{1}{2} (\partial_\mu \phi)^2 = \int d^dx \sqrt{g} \, \frac{1}{2} \partial_\mu \phi \, \partial_\nu \phi \, g^{\mu\nu}$$

The metric in spherical coordinates is

$$ds^2 = dr^2 + r^2 \underbrace{d\Omega^2_{d-1}}_{\text{metric on a } d-1 \text{ dimensional sphere } S^{d-1}}$$

Change coordinates: $r = e^\sigma$

$$ds^2 = e^{2\sigma} \underbrace{\left(d\sigma^2 + d\Omega^2_{d-1} \right)}_{\text{this is the metric on the cylinder, by the way...}} \qquad (1)$$

Notation: i, j = coordinates on the sphere
μ, ν = all coordinates

$$\to \quad \sqrt{g} = \sqrt{\det g_{\mu\nu}} = e^{d\sigma}\sqrt{\det g_{ij}}$$

metric on the sphere $d\Omega^2_{d-1} = g_{ij}\, d\theta^i d\theta^j$

$$\to \quad g^{\mu\nu} = e^{-2\sigma}\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & g^{ij} \\ 0 & \end{pmatrix}$$

So

$$S_E = \frac{1}{2}\int_{-\infty}^{+\infty} d\sigma \underbrace{\int d^{d-1}\theta \sqrt{\det g_{ij}}}_{\equiv d\Omega}\; e^{(d-2)\phi}\left((\partial_\sigma\phi)^2 + \partial_i\phi\,\partial_j\phi\, g^{ij}\right)$$

$$\phi = e^{-\frac{d-2}{2}\sigma}\chi \rightsquigarrow \partial_\sigma\phi = \frac{2-d}{2}\, e^{-\frac{d-2}{2}}\chi + e^{-\frac{d-2}{2}\sigma}\partial_\sigma\chi$$

$$S_E = \frac{1}{2}\int d\sigma\, d\Omega \left((\partial_\sigma\chi)^2 + \partial_i\chi\,\partial^i\chi + \left(\frac{d-2}{2}\right)^2\chi^2 + \cancel{\text{total derivative}}\right)$$

OBSERVATION Compare this computation with what you know about Weyl symmetry of the action of a free scalar:

The action $S_\xi[\phi, g] = \int\sqrt{g}\left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\xi R\phi^2\right)$

is Weyl invariant for the right choice of ξ:

$$S_\xi[\Omega^{-\Delta}\phi, \Omega^2 g] = S_\xi[\phi, g]$$

it is of course also diffeomorfism invariant, and it reduces to S_E in flat space.

Now choose $\Omega(x) = e^{-\sigma}$, $\chi \equiv \Omega^{-\Delta}\phi = e^{\frac{d-2}{2}\sigma}\phi$
and $g_{\mu\nu} = \delta_{\mu\nu}$:

$$S_\xi[\chi, e^{-2\sigma}\delta_{\mu\nu}] = S_E \qquad (2)$$

let's do a diffeo now: $S_\xi[\phi', g'] = S[\phi, g]$

where: $\begin{cases} g'_{\mu\nu}(x') = \dfrac{\partial x^\lambda}{\partial x'^\mu}\dfrac{\partial x^\sigma}{\partial x'^\nu} g_{\lambda\sigma}(x) \\ \phi'(x') = \phi(x) \end{cases}$

If we choose $r = e^\sigma$ as a diffeo, and $g_{\mu\nu} = e^{-2\sigma}\delta_{\mu\nu}$,
then $g'_{\mu\nu}dx'^\mu dx'^\nu = d\sigma^2 + d\Omega^2_{d-1}$
(this immediately follows from (1) above)

And we get from (2):

$$S_E = S_\xi[\chi', g_{cylinder}] =$$

$$= \int d\sigma d\Omega \left[(\partial_\sigma \chi'(\sigma,\theta^i))^2 + \partial_i \chi' \partial^i \chi + \right.$$

$$\left. + \frac{1}{2}\xi R_{cylinder} \chi'^2 \right]$$

Notice that (σ, θ^i) are our primed coordinates so $\chi'(\sigma, r) = \chi$

So:

$$S_E = \int d\sigma d\Omega \left((\partial_\sigma \chi)^2 + \partial_i \chi \partial^i \chi + \frac{1}{2} \xi R_{cyl} \chi^2 \right)$$

So the result we got could actually be found by evaluating R on the cylinder $S^{d-1} \times \mathbb{R}$.

This also justifies the result of point 5 of the exercise

2. The e.o.m. are:

$$\partial_\sigma^2 \chi + \Box_{S^{d-1}} \chi - \Delta^2 \chi = 0 \qquad \Delta = \frac{d-2}{2}$$

$\Box_{S^{d-1}}$ is the Laplacian on the sphere

(The general expression is $\Box = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, g^{\mu\nu} \partial_\nu \right)$)

From now on we work in $d=3$.

$$\chi(\sigma,\theta^i) = \sum_{\ell m} b_{\ell m}(\sigma)\, Y_{\ell m}(\theta^i)$$

Plug into the eom: $\quad \partial_\sigma^2 b_{\ell m} = \left(\ell+\frac{1}{2}\right)^2 b_{\ell m} \equiv \omega^2 b_{\ell m}$

$$\rightarrow \qquad b_{\ell m} = b^{(+)}_{\ell m}\, e^{\omega\sigma} + b^{(-)}_{\ell m}\, e^{-\omega\sigma}$$

The hermiticity property on the cylinder is:

$$\chi^+(\sigma,\theta^i) = \chi(-\sigma,\theta^i)$$

(You know this from the lecture on quantization on the cylinder)

Let's rewrite:

$$\chi = \sum_\ell \sum_{m=-\ell}^{\ell} \left(b^{(-)}_{\ell m}\, e^{-\omega\sigma} Y_{\ell m} + b^{(+)}_{\ell m} Y_{\ell m}\right) =$$

$$= \sum_\ell \sum_{m=-\ell}^{\ell} \Big(\underbrace{b^{(-)}_{\ell\, -m} (-1)^m}_{\equiv\, a^{(-)}_{\ell m}/\sqrt{2\omega}}\, e^{-\omega\sigma}\, Y^*_{\ell m} + \underbrace{b^{(+)}_{\ell m}}_{\equiv\, a^{(+)}_{\ell m}/\sqrt{2\omega}}\, Y_{\ell m} \Big)$$

(Definition familiar from harmonic oscillator)

Hermitricity then implies:

$$\left[a^{(-)}_{em}\right]^{+} = a^{(+)\,*}_{em}$$

Now let's expand the Hamiltonian in modes:

$$\chi = \frac{1}{\sqrt{2\omega}} \sum_{em} a^{(-)}_{em} e^{-\omega\sigma} Y^{*}_{em} + a^{(+)}_{em} e^{\omega\sigma} Y_{em}$$

$$\partial_\sigma \chi = \sqrt{\frac{\omega}{2}} \sum_{em} - a^{(-)}_{em} e^{-\omega\sigma} Y^{*}_{em} + a^{(+)}_{em} e^{\omega\sigma} Y_{em}$$

⊗ Using $\int d\Omega\, Y_{em} Y^{*}_{e'm'} = \delta_{ee'}\delta_{mm'}$ and $Y_{e-m} = (-1)^m Y^{*}_{em}$ we get rid of the Y_{em} in products like $(\partial_\sigma\chi)^2$ and χ^2

⊗ Also $\partial_i \chi \partial^i \chi$ can be integrated by parts:

$$\int d\Omega\, \frac{1}{2} \partial_i \chi \partial^i \chi = - \int d\Omega\, \frac{1}{2} \chi \Box \chi$$

$$= - \sum_{em} \frac{1}{2} \left(-e(e+1)\right)\left(a^{(-)}_{em} a^{(+)}_{em} + a^{(+)}_{em} a^{(-)}_{em}\right) \frac{1}{2\omega}$$

All together, $a^{(-)}a^{(-)}$ and $a^{(+)}a^{(+)}$ cancel:

$$H = \frac{1}{2} \sum_{em} \left\{ - \frac{\omega}{2}\left(- a^{(-)}_{em} a^{(+)}_{em} - a^{(+)}_{em} a^{(-)}_{em}\right) + \frac{e(e+1)}{2\omega}(\text{same}) + \right.$$

$$\left. + \frac{1}{4}\frac{1}{2\omega}(\text{same}) \right\} =$$

$$= \frac{1}{2} \sum_{em} \omega \left(a^{(-)}_{em} a^{(+)}_{em} + a^{(+)}_{em} a^{(-)}_{em} \right) \quad \blacksquare$$

3. Using $\sum_{em} Y^*_{em}(\theta') Y_{em}(\theta'') = \delta^2(\theta - \theta')$

(I am defining a curved space version of the δ-function which contains the right denominator:

$$\delta^d(x) = \frac{1}{\sqrt{g}} \delta(x^1) \ldots \delta(x^d)$$
)

we see that only $Y^*_{em} Y_{em}$ gives the right δ-function.

So:

$$[a^{(-)}_{em}, a^{(-)}_{e'm'}] = 0$$

$$[a^{(-)}_{em}, a^{(+)}_{e'm'}] = k\, \delta_{ee'} \delta_{mm'} .$$

Fix k:

$$[\chi(\sigma,\theta), \partial_\sigma \chi(\sigma',\theta')] = \frac{1}{2} \sum_{\substack{em \\ e'm'}} \Big[a^{(-)}_{em} e^{-\omega\sigma} Y^*_{em} + a^{(+)}_{em} e^{\omega\sigma} Y_{em},$$

$$- a^{(-)}_{e'm'} e^{-\omega\sigma} Y^*_{e'm'} + a^{(+)}_{e'm'} e^{\omega\sigma} Y_{e'm'} \Big] =$$

$$= \frac{1}{2} \sum_{em} -(-k) Y_{em}(\theta) Y^*_{em}(\theta') + k Y^*_{em}(\theta) Y_{em}(\theta')$$

$$= k\,\delta^2(\theta-\theta') \longrightarrow \boxed{k=1}$$

NOTE: In fact, one could fix the sign of $[\chi, \partial_\sigma \chi]$ based on $[a, a^+] = 1$. In fact:

1) ϕ should be regular in the origin $(\sigma = -\infty)$ when put inside a correlator.

2) Time ordering in σ (aka radial ordering) puts on the right the field with smaller r:

$$\langle 0 | R\left(\phi(r,\theta')\,\phi(0,\theta'')\right)|0\rangle = \langle 0|\phi(r,\theta')\,\phi(0,\theta')|0\rangle$$

if $r > 0$

3) $\Rightarrow$ $a^{(-)}_{em}|0\rangle = 0$ otherwise the correlator is infinite

4) So $a^{(-)}_{em}$ is the annihilation operator, and the Hamiltonian must be:

$$H = a^+ a + \text{c-number} = a^{(+)}a^{(-)} + \text{c-number}$$

and $[a, a^+] = 1 \rightsquigarrow [a^{(-)}, a^{(+)}] = 1$

(Indeed recall: $\langle 0|a\,a^+|0\rangle = \|a^+|0\rangle\|^2 > 0$
$= \langle 0|[a, a^+]|0\rangle$

so $[a, a^+] > 0$)

5) Going backwards, this fixes the sign of $[\chi, \partial_\tau \chi]$

So, disregarding the usual infinite vacuum energy, we get the spectrum :

$$\epsilon_{\ell_1 \cdots \ell_n} = \left(\ell_1 + \frac{1}{2}\right) + \cdots + \left(\ell_n + \frac{1}{2}\right)$$

4. Local operators are constructed by taking products of derivatives of fields :

$$\mathcal{O}_{\mu_1 \cdots \mu_L} = :\partial_{\mu_1} \cdots \partial_{\mu_{\ell_1}} \phi \, \partial_{\mu_{\ell_1+1}} \cdots \partial_{\mu_{\ell_1+\ell_2}} \phi \cdots : \quad \leftarrow n \text{ fields}$$

$$\sum_i^n \ell_i = L$$

Each derivative is a symmetric traceless tensor of $SO(3)$
It's traceless because $\Box \phi = 0$ by the eom.

$\Rightarrow$ it forms an IRREP of $SO(3)$, whose elements are labeled by (ℓ, m).

Want to check?

$$\begin{pmatrix} \text{\# independent components} \\ \text{in a symmetric tensor of} \\ \text{rank } \ell \text{ in } d \text{ dimensions} \end{pmatrix} \underbrace{=}_{\text{Try to find this!}} \binom{\ell+d-1}{d-1}$$

Binomial

$$\begin{pmatrix} \text{\# indip. components if the} \\ \text{tensor is traceless} \end{pmatrix} = \binom{\ell+d-1}{d-1} - \binom{\ell-2+d-1}{d-1}$$

$$= \binom{\ell+d-1}{d-1} - \binom{\ell+d-3}{d-1}$$

because we need to subtract away the components of the tensor of rank $\ell-2$ obtained by taking a trace.

Plug $d=3$, $\binom{\ell+2}{2} - \binom{\ell}{2} = 2\ell+1 = \sum_{m=-\ell}^{\ell} 1$ ✓

So states $a^{(+)}_{\ell_1 m_1} \dots a^{(+)}_{\ell_n m_n} |0\rangle$ are in 1 to 1 corresp. with local operators.

And energy of these states = scaling dimension of the associated local operator

Indeed $\Delta_\phi = \frac{d-2}{2} = \frac{1}{2}$ and $[\partial_\mu] = 1$,

$\boxed{\text{This is STATE-OPERATOR correspondence!}}$

5. Spectrum of the dilatation current: $J_\mu = -T_{\mu\nu}x^\nu$

- We need to compute $x^\mu x^\nu T_{\mu\nu}$

Use: $x^\mu \partial_\mu = r\partial_r$ and $x^\mu x^\nu \partial_\mu \partial_\nu = r\partial_r(r\partial_r) - r\partial_r$

$$x^\mu x^\nu T_{\mu\nu} = (r\partial_r\phi)^2 - \frac{1}{2}r^2(\partial\phi)^2 - \xi\left[r\partial_r(r\partial_r - 1)\phi^2 - 2r^2(\partial\phi)^2\right]$$

$$r = e^\sigma \quad \phi = e^{-\underset{\uparrow}{\Delta}\sigma}\chi \quad ds^2 = e^{2\sigma}(d\sigma^2 + d\Omega^2)$$

After a while: $\frac{d-2}{2}$

$$x^\mu x^\nu T_{\mu\nu} = \frac{1}{2}e^{-2\Delta\sigma}\Big\{\Delta(\Delta - 4\xi - 4\Delta\xi)\chi^2 + (\partial_\sigma\chi)^2 +$$
$$+ (-2\Delta + 4\xi + 8\Delta\xi)\chi\partial_\sigma\chi - 4\xi\chi\partial_\sigma^2\chi + (-1+4\xi)\partial_i\chi\partial^i\chi\Big\}$$

Eom: $\partial_\sigma^2\chi = \Delta\chi - \Box_{S^{d-1}}\chi$

$x^\mu x^\nu T_{\mu\nu}$ will be integrated in $d\Omega$, so let's allow integration by parts:

$$\chi\partial_\sigma^2\chi \to \Delta\chi^2 + \partial_i\chi\partial^i\chi$$

We got:

$$x^\mu x^\nu T_{\mu\nu} \to \frac{1}{2}e^{-2\Delta\sigma}\Big\{-\partial_i\chi\partial^i\chi + (-4\Delta^2\xi + \Delta(\Delta - 4\xi - 4\Delta\xi))\chi^2 +$$

$$+ \left(8\Delta\xi + 4\xi - 2\Delta\right)\chi\partial_\sigma\chi + (\partial_\sigma\chi)^2 \Big\}$$

⊗ Observation: the mode expansion of $\chi\partial_\sigma\chi$ only shifts the vacuum:

$$\int d\Omega\, \chi\partial_\sigma\chi \sim \sum_{\ell m}\left(a^{(-)}_{\ell m}a^{(+)}_{\ell m} - a^{(+)}_{\ell m}a^{(-)}_{\ell m}\right) = \sum_{\ell m} 1$$

so we drop it.

We are left with:

$$\int d\Omega\, r^{d-1}\frac{x^\mu}{r}J_\mu = -\int d\Omega\, r^{d-2}\, x^\mu x^\nu\, T_{\mu\nu} =$$

$$= \dots = H \qquad \text{once we set} \qquad \xi = \frac{d-2}{4(d-1)}$$

So we get that the dilatation operator is the Hamiltonian on the cylinder. Indeed recall:

$$D = -x^\mu\partial_\mu$$

and to each conformal Killing vector v^μ we can associate the current

$$v^\mu T_{\mu\nu}$$

indeed: $$\nabla^\nu\left(v^\mu T_{\mu\nu}\right) = \nabla^\nu v^\mu\, T_{\mu\nu} + v^\mu \underbrace{\cancel{\nabla_\nu T_{\mu\nu}}}_{0 \text{ conservation}}$$

$$= \nabla^{(\nu} v^{\mu)} \, T_{\mu\nu}$$

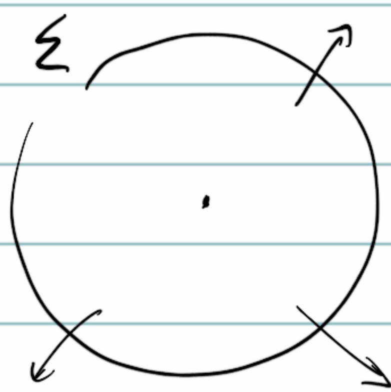
symmetry of $T_{\mu\nu}$

but recall $\nabla_{(\mu} v_{\nu)} \propto g_{\mu\nu}$ for a conf. killing vector

$$\propto g^{\mu\nu} T_{\mu\nu} = 0 \quad \text{from conformal invariance.}$$

So $\quad j_\nu = - x^\mu T_{\mu\nu}$ generates dilatations,

The integral $\quad \int d\Omega \, r^{d-1} \frac{x^\mu}{r} j_\mu = \int d\Sigma^\mu j_\mu$



is just the flux of the current on a sphere:
it measures the charge of an operator at the origin,
i.e. its scaling dimension

$\boxed{\text{On the density of states}}$. ex. 2.7.1 in the notes.

We want to invert the spectral representation of the two-point function to find $\rho(\epsilon, \vec{k})$:

$$\int \frac{d\epsilon\, d\vec{k}}{(2\pi)^d} e^{-\epsilon\tau + i\vec{k}\cdot\vec{x}} \rho(\epsilon, \vec{k}) = \frac{1}{(\tau^2 + \vec{x}^2)^\Delta} \qquad \tau > 0$$

Inverse Fourier transform, and use $\frac{1}{A^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{ds}{s} s^\alpha e^{-sA}$

$$\int \frac{d\epsilon}{2\pi} e^{-\epsilon\tau} \rho(\epsilon, \vec{k}) = \int d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}} \frac{1}{\Gamma(\Delta)} \int_0^\infty \frac{ds}{s} s^\Delta e^{-s\tau^2 - s\vec{x}^2}$$

The integral in $d\vec{x}$ is Gaussian:

$$= \frac{1}{\Gamma(\Delta)} \int_0^\infty \frac{ds}{s} s^\Delta \left(\frac{\pi}{s}\right)^{\frac{d-1}{2}} e^{-s\tau^2 - \frac{\vec{k}^2}{4s}}$$

We need the Laplace transform, and it is convenient to first change variable:

$$s \to s/\tau$$

$$= \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\Delta)} \tau^{\frac{d-1}{2} - \Delta} \int_0^\infty ds/s\; s^{\Delta - \frac{d-1}{2}} \underbrace{e^{-\tau\left(s + \frac{k^2}{4s}\right)}}_{}$$

We want to inverse-Laplace transform this

Indeed recall the definition of a Laplace transform:

$$\hat{f}(\tau) \equiv \mathcal{L}[f](\tau) = \int_0^\infty dE\, e^{-E\tau} f(E)$$

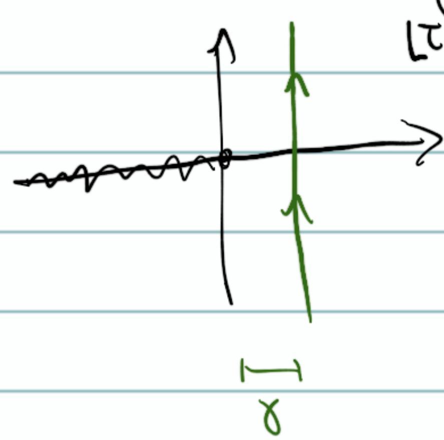
Inverse: $$\mathcal{L}^{-1}[\hat{f}](E) = f(E) = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{+i\infty+\gamma} d\tau\, e^{E\tau}\,\hat{f}(\tau)$$

The contour goes on the right of all singularities.

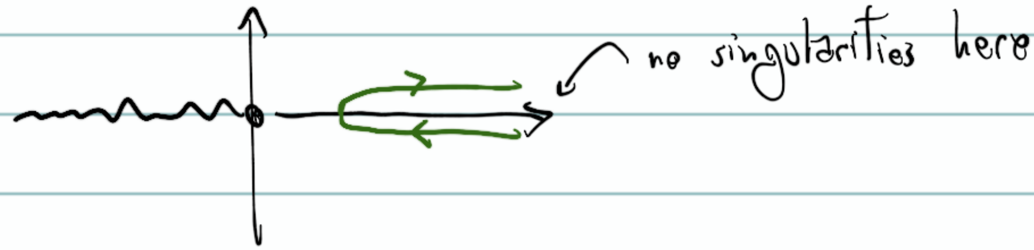
So we need:

$$I = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{+i\infty+\gamma} d\tau\, \tau^{\alpha} e^{(E-\beta)\tau} \qquad \begin{cases} \alpha = \frac{d-1}{2} - \Delta \\ \beta = s + \frac{k^2}{4s} \end{cases}$$

τ^{α} has a cut on the negative real axis:

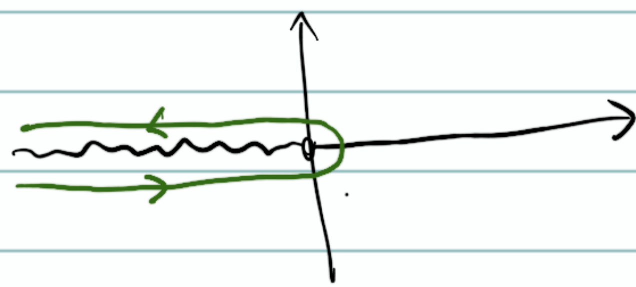


Now if $E - \beta < 0$ we need to pull the contour to the right, 'cause $e^{(E-\beta)\tau}$ has to be suppressed:

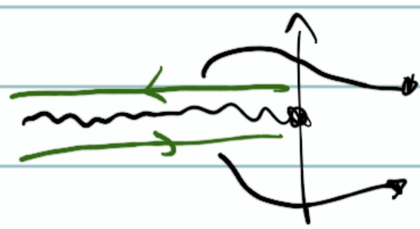


so we get zero.

If $E-\beta > 0$ we pull to the left:



So we integrate the discontinuity of τ^α:



$\operatorname{Arg}\tau = i\pi \rightsquigarrow \tau^\alpha = |\tau|^\alpha e^{i\pi\alpha}$

$\operatorname{Arg}\tau = -i\pi \rightsquigarrow \tau^\alpha = |\tau|^\alpha e^{-i\pi\alpha}$

So

$$I = \frac{1}{2\pi i}\int_0^{-\infty} d\tau\, |\tau|^\alpha \left(e^{i\pi\alpha} - e^{-i\pi\alpha}\right) e^{(E-\beta)\tau}\, \Theta(E-\beta)$$

$$= -\frac{1}{\pi}\sin\pi\alpha \int_0^{\infty} d\tau\, \tau^\alpha e^{-(E-\beta)\tau}\, \Theta(E-\beta)$$

$$= -\frac{1}{\pi}\sin\pi\alpha\, (E-\beta)^{-1-\alpha}\, \Gamma(\alpha+1)\, \Theta(E-\beta)$$

Finally recall: $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin\pi z}$ choose $z = -\alpha$

$$= \frac{1}{\Gamma(-\alpha)} (E-\beta)^{-1-\alpha}\, \Theta(E-\beta)$$

going back:

$$\rho(E,\vec{k}) = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\Delta-\frac{d-1}{2}\right)\Gamma(\Delta)} \int_0^\infty ds\, s^{\Delta-\frac{d+1}{2}} \left(E-s-\frac{k^2}{4s}\right)^{\Delta-\frac{d+1}{2}} \times$$

$$\times\, \Theta\left(E-s-\frac{k^2}{4s}\right)$$

One integral left:

$$= \frac{2\pi^{d+1/2}}{\Gamma\left(\Delta-\frac{d-1}{2}\right)\Gamma(\Delta)} \int_0^\infty ds \left(sE-s^2-\frac{k^2}{4}\right)^{\Delta-\frac{d+1}{2}}$$

$$\Theta\left(Es-s^2-\frac{k^2}{4s}\right)$$

$s>0$ so this shift does nothing

$$sE-s^2-\frac{k^2}{4} = (s_+-s)(s-s_-)$$

$$s_\pm = \frac{E\pm\sqrt{E^2-k^2}}{2}$$

Lorentzian dispersion relation is coming out of a Euclidean computation!

$$= \frac{2\pi^{(d+1)/2}}{\Gamma\left(\Delta-\frac{d-1}{2}\right)\Gamma(\Delta)} \int_{s_-}^{s_+} ds \left[(s_+-s)(s-s_-)\right]^{\Delta-\frac{d+1}{2}}\, \Theta(E-|k|)$$

Notice that $s_- > 0$ for $E > 0$, and also the $\theta(E-|k|)$ arises because s_+ and s_- are not real if $E^2-k^2<0$ and they are both negative iff $E<0$.

$$= \frac{2\pi^{\frac{(d+1)}{2}}}{\Gamma\left(\Delta-\frac{d-1}{2}\right)\Gamma(\Delta)} (s_+-s_-)^{2\Delta-d} \underbrace{\int_0^1 dx\left((1-x)x\right)^{\Delta-\frac{d+1}{2}}}_{=\frac{\Gamma\left(\Delta-\frac{d-1}{2}\right)^2}{\Gamma(2\Delta-d+1)}} \theta(E-|k|)$$

$$= \frac{2\pi^{\frac{d+1}{2}}\,\Gamma\left(\Delta-\frac{d-1}{2}\right)}{\Gamma(\Delta)\,\Gamma(2\Delta-d+1)} \left(E^2-k^2\right)^{\Delta-d/2} \theta(E-|k|)$$

$$\frac{\Gamma(2\alpha)}{\Gamma(\alpha)} = \frac{2^{2\alpha-1}}{\sqrt{\pi}}\,\Gamma\left(\alpha+\frac{1}{2}\right) \qquad \alpha = \Delta-\frac{d-1}{2}$$

$$\boxed{\rho(E,k) = \frac{2^{-2\Delta+d+1}\,\pi^{\frac{d}{2}+1}}{\Gamma(\Delta)\,\Gamma\left(\Delta-\frac{d}{2}+1\right)} \left(E^2-k^2\right)^{\Delta-\frac{d}{2}} \theta(E-|k|)}$$

typo in the notes of the course?

- Reflection positivity implies $\rho > 0 \Rightarrow \Delta - \frac{d}{2} + 1 \geq 0$
we got back the unitarity bound found in a previous exercise.

- When $\Delta = \frac{d}{2} - 1$ we need to use a limiting procedure.
The prefactor vanishes, but $(\epsilon^2 - k^2)^{-1} \to \infty$ when $\epsilon^2 = k^2$ ...

Now recall that a δ-function is defined by the property:

$$\int_{-\infty}^{+\infty} dx\, f(x)\, \delta(x) = f(0)$$

for every f which goes to zero "fast enough" at $\pm\infty$.

Consider:

$$\lim_{\varepsilon \to 0} \varepsilon \int_{-\infty}^{+\infty} dx\, f(x)\, x^{\varepsilon - 1}\, \theta(x) =$$

$$= \lim_{\varepsilon \to 0} \varepsilon \int_0^{\infty} dx\, f(x)\, x^{\varepsilon - 1} \underset{\text{by parts}}{=} \lim_{\varepsilon \to 0} \left(x^{\varepsilon} f(x) \Big]_0^{\infty} - \int_0^{\infty} dx\, x^{\varepsilon} f'(x) \right)$$

$$= \lim_{\varepsilon \to 0} \left(- \int_0^{\infty} dx\, x^{\varepsilon} f'(x) \right)$$

Now since the integral converges in $\varepsilon = 0$ we can switch limit and integral

$$= -\int_0^\infty f'(x)\,dx = f(0)$$

we conclude:

$$\lim_{\varepsilon\to 0} \varepsilon\, x^{\varepsilon-1}\,\Theta(x) = \delta(x)$$

limit must be taken after integrating against a test function: it is called a weak limit.

We conclude:

$$\rho(E,\vec{k})\Big|_{\Delta=\frac{d}{2}-1} = \frac{8\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d}{2}-1\right)}\,\delta(E^2-k^2)$$

Recall that $\rho(E,\vec{k}) = \sum_\alpha \left|\langle \mathcal{O}(\tau=0,\vec{x}=0)\,|\,E,\vec{k},\alpha\rangle\right|^2$

where α distinguishes states with the same $(E,\vec{k})$

We see that in this case $|\mathcal{O}(\tau,\vec{x})\rangle$ overlaps only with states containing a single massless particle: $E^2=k^2$.

$\Rightarrow$ The free theory contains massless particles: we knew that!

But if the CFT does NOT contain an operator of dimension $\Delta=\frac{d}{2}-1$, no primary overlaps w/ single particle states: the theory has NO particles!

(An operator with spin might still overlap with massless particles with spin though: a free boson is not the only free theory...)

- Now consider the case $\Delta = d-2$ and compare with the phase space of two particles:

$$\phi_{2p.}(E,\vec{k}) = \int dE_1 d\vec{k}_1\, dE_2 d\vec{k}_2\; \Theta(E_1)\Theta(E_2)\,\delta(E_1^2-\vec{k}_1^2)\,\delta(E_2^2-\vec{k}_2^2) \times$$
$$\delta(E_1+E_2-E)\,\delta^{d-1}(\vec{k}_1+\vec{k}_2-\vec{k})$$

$$= \int dE_1 d\vec{k}_1\; \Theta(E_1)\,\Theta(E-E_1)\,\delta(E_1^2-\vec{k}_1^2)\,\delta\left((E-E_1)^2-(\vec{k}-\vec{k}_1)^2\right)$$

Recall: $\displaystyle\int dE_1\, \Theta(E_1)\,\delta(E_1^2-\vec{k}_1^2)\, f(E_1) = \frac{1}{2|\vec{k}_1|} f(|\vec{k}_1|)$

$$= \int \frac{d\vec{k}_1}{2|\vec{k}_1|}\, \Theta(E-|\vec{k}_1|)\,\delta\left((E-|\vec{k}_1|)^2-(\vec{k}-\vec{k}_1)^2\right)$$

We can get a quick answer by noticing that $\phi(E,\vec{k})$ is boost invariant: it can only be a function of E^2-k^2. Furthermore, $\Theta(E-|k_1|)$ implies that there is no support if $E<0$ $\rightarrow$ by Lorentz invariance then there is support only in the interior of the future lightcone (if ϕ was non-zero for a space like point with $E>0$, it would be non zero for some $E<0$

as well). So $\phi(\vec{k},\epsilon) = \phi(\vec{k},\epsilon)\,\Theta(\epsilon - |k|)$

Then let's set $\vec{k}=0$ and reconstruct in the end:

$$\xrightarrow[\vec{k}=0]{} \int \frac{d\vec{k}_1}{2|k_1|}\, \Theta(\epsilon - |k_1|)\, \delta\big(\epsilon(\epsilon - 2|k_1|)\big)$$

$$= \int d\Omega_{d-1} \int_0^{\epsilon} dk_1\, k_1^{d-3}\, \frac{1}{2\epsilon}\, \delta\!\left(k_1 - \frac{\epsilon}{2}\right)$$

$$= \underbrace{\Omega_{d-1}}_{\text{solid angle in } d-1 \text{ dim.}}\, 2^{3-d}\, \epsilon^{d-4}$$

$$\xrightarrow[\text{turn on } \vec{k}]{} = \Omega_{d-2}\, 2^{3-d}\, (\epsilon^2 - k^2)^{\frac{d}{2}-2}\, \Theta(\epsilon - |k|)$$

from reasoning above.

So: $$\rho(\vec{k},\epsilon)\Big|_{\Delta = d-2} \propto \phi_{2p}(\vec{k},\epsilon)$$

This is indeed what we expect from an operator like ϕ^2 that generates 2 particle states:

$$\rho(\vec{k},\epsilon) = \sum_{\alpha} \langle \phi^2(0) | k, \alpha \rangle \langle \alpha, k | \phi^2(0) \rangle \qquad k = (\vec{k}, \epsilon)$$

α = counts the states with the same momentum k

In a theory with particles, $\sum_\alpha$ can be split into the phase space of n-particles:

$$\sum_\alpha |k,\alpha\rangle\langle k,\alpha| = \underbrace{\sum_n}_{n = \#\text{ of particles}} \underbrace{\int d\phi_n(q_1, \dots q_n)}_{\substack{\text{phase space of}\\ \text{n-particles}\\ (q_i \text{ are integrated})}} \underbrace{\delta\Big(\sum_i q_i - k\Big)}_{\substack{\text{restriction to}\\ \text{states of total}\\ \text{momentum } k.}} \underbrace{|n,q_i\rangle\langle n,q_i|}_{\substack{\text{n particle states}\\ \text{with momenta } q_i}}$$

Now $|\phi^2(0)\rangle = \int dk'\, dq'\, a^\dagger(q')\, a^\dagger(k'-q')|0\rangle + \dots$

pieces with $aa^\dagger$

only has overlap with 2 particle states:

$$\langle 0|\, a(k)\, a(q-k)\, a^\dagger(q')\, a^\dagger(k'-q')|0\rangle$$
$$\propto \delta^d(q-k-q')\,\delta^d(k-k'+q') + \delta^d(k-q')\,\delta^d(q-k-k'+q')$$

$$\Rightarrow \quad \langle 2, q_i|\, \phi^2(0)\rangle = \text{constant}$$

So: $$\rho(\vec{k},\epsilon) = |\langle 2, q_i | \phi^2(0) \rangle|^2 \int d\phi_2(q_i)\, \delta^d\left(\sum_i q_i - k\right)$$

$$= \text{constant} \times \phi_{2p}(k, \vec{\epsilon})$$

exactly as found from direct computation. ∎